Parking Enforcement: How to optimally choose the citation fine and level-of-enforcement

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Abstract

A parking enforcement policy, in its simplest form, is comprised of a citation fine and a level-of-enforcement. The citation fine is the penalty paid by illegally parked vehicles that get a parking ticket and the level-of-enforcement is the number of enforcement units (e.g., cameras or on-foot officers) deployed in a region to find illegally parked vehicles. In this paper, we investigate how to optimally devise a parking enforcement policy to maximize social welfare and profit. To find the optimal policy, we first show that the conventional inspection-game methodology, commonly used for modelling enforcement environments, cannot be applied to parking enforcement; the inspection-game does not realistically represent the searching process where enforcement units seek out illegally parked vehicles. Given that it takes time to find and cite each illegally parked vehicle, there is friction present in the searching process. To quantify the friction, we use the bilateral-search-and-meet function and we characterize key factors of illegal parking behavior such as parking dwell time, probability of parking illegally, citation probability, and rate of citations. Using these factors, we present an equilibrium model of illegal parking where each driver first decides to park legally or illegally and next chooses the parking dwell time. We prove that the equilibrium exists and is unique. The model yields several logical and some non-intuitive insights: (i) the citation probability increases with the illegal dwell time because vehicles that are parked for a long time are more susceptible to getting a citation, (ii) the citation probability decreases with the number of illegally parked vehicles, (iii) vehicles are more likely to park illegally when their dwell time is short, and (iv) the citation fine and the level-of-enforcement are lowered as the enforcement technology becomes more efficient.

Keywords: Parking; Enforcement; Bilateral Meeting; Inspection Game; Equilibrium

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1 Introduction

Illegal parking leads to adverse societal impacts such as reduced traffic speeds (Han et al., 2005), loss of revenue from legal parking (Cullinane and Polak, 1992), and more accidents caused by safety violations (Conway et al., 2013). It is estimated that illegal parking causes 47 million vehicle-hours of delay each year in the United States, which makes illegal parking the third leading cause of delay behind construction and crashes (Han et al., 2005). In response to these detrimental consequences, parking enforcement policies are implemented to hinder illegal parking. A parking enforcement policy, in its simplest form, is comprised of choosing a citation fine and the level-of-enforcement. The citation fine is the penalty paid by illegally parked vehicles that get caught by an enforcement unit (e.g. on-foot officers or cameras) and the level-of-enforcement is the number of enforcement units deployed in a region to find illegal vehicles. In this paper, we investigate how to optimally develop an enforcement policy that helps cities better manage illegal parking behavior.

The impact of an enforcement policy on illegal parking behavior is conceptually illustrated in Fig. 1. Illegal parking behavior is influenced by the citation probability and the citation fine; drivers are less likely to park illegally when the citation probability is high and the citation fine is large. While the fine is directly stipulated by parking authorities, derivation and analysis of the citation probability is more intricate. When a vehicle is parked illegally for a long time, it has a higher citation probability because of its longer exposure to getting caught by an enforcement unit. Although it is acknowledged that the citation probability is sensitive to dwell time, most studies assume the citation probability to be a fixed parameter. The drawback of this assumption is that the proposed models are limited in representing dwell time which is a critical component of illegal parking behavior. In addition to dwell time, the citation probability is also sensitive to the level-of-enforcement; the citation probability is higher when more enforcement units are deployed to find illegal vehicles. The outcome of the enforcement policy is the number of vehicles that park illegally which can be controlled by parking authorities when the optimal policy is implemented.

Parking enforcement policies are commonly developed to achieve two objectives. The first objective is to improve social welfare by mitigating the negative effects of illegal parking (Cullinane and Polak, 1992) and the second objective is to raise profits generated from legal and/or illegal parking. The citation profits are substantial in many cities. In 2013, New York City, Los Angeles, and Chicago each generated 534, 250, and 176 million dollars, respectively. In some instances, target profits are defined annually and policies are devised to reach them. We investigate how parking authorities can optimize each of these objectives while taking into account the reactive illegal parking behavior of drivers.

This paper is organized as follows. A review of research on illegal parking is presented in Section 2 and the gaps in the literature are highlighted. A choice model of parking legally or illegally is presented in Section 3. The equilibrium that arises from this choice model is presented in Section 4. Properties of the equilibrium model are investigated in Section 5. An optimal parking enforcement policy is derived for social welfare and profit maximization in Section 6. A numerical experiment is conducted and analyzed in Section 7. Conclusions are presented in Section 8.

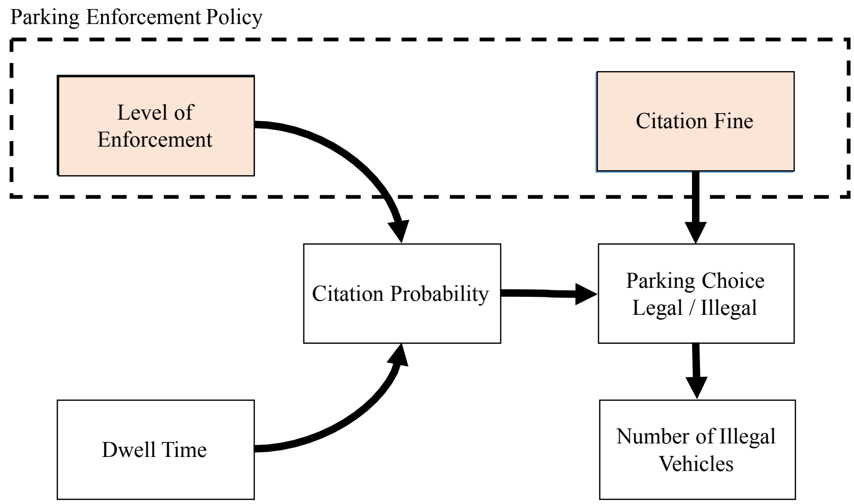

Fig. 1. Impact of parking enforcement policies on illegal parking behavior.

2 Literature review

2.1 Parking enforcement and illegal parking behavior

Given that parking enforcement is a policy-maker's response to illegal parking, it is vital that there is a sound understanding of the patterns and causes of illegal parking. Many studies have investigated the causes of illegal parking by focusing on aspects such as illegal parking behavior in central business districts (Brown, 1983), impact of the parking citation fine on public transportation ridership (Auchincloss et al., 2014), and illegal parking behavior of commercial vehicles (Wang and Gogineni, 2015; Wenneman et al., 2015). Most of these studies, however, are empirical investigations of the factors that influence illegal parking behavior. While identifying these influential factors brings us a step closer to understanding illegal parking behavior, there is still a need for models that can quantitatively assess the impact of enforcement policies. In a recent review of the literature on parking, Inci (2014) emphasizes the immediate need for theoretical and analytical models of parking enforcement that take into account illegal parking behavior.

There are currently only a handful of studies that develop analytical models of illegal parking and parking enforcement. Petiot (2004) presents a parking model where each driver makes a binary choice of parking legally or illegally based on the utility obtained from each choice. Petiot's (2004) model, which is an extension of the model of Arnott and Rowse (1999), captures the impact of the citation fine on illegal parking behavior but is not sensitive to parking duration or the level-of-enforcement. Accounting for parking duration and level-of-enforcement is non-trivial as these two factors strongly influence the citation probability. A larger citation probability reduces the utility of illegal parking which in turn influences a driver's choice of parking legally or illegally. Lack of a representation of the citation probability, illegal dwell time, and level-of-enforcement is evident in other theoretical studies of illegal parking such as Elliot and Wright (1982), Cullinane (1993), and Thomson and Richardson (1998). In this paper, we focus on developing an analytical model of illegal parking with an explicit representation of these important factors and we investigate the changes in illegal parking behavior with respect to the implemented enforcement policy. To proceed, we first review in the next section the topic of inspection games as it is widely used to model the impact of enforcement on any type of illegal behavior.

2.2 The inspection game

The inspection game is a classical methodology for modelling environments where enforcement units, known as inspectors, seek potential parking violators called inspectees (Ferguson and Mlolidakis, 1998; Avenhaus and Canty, 2005; Avenhaus and Canty, 2012). Examples of the inspection game include ticket-inspection by barrier-free transit providers (Sasaki, 2014; Barabino et al., 2014; Barabino et al., 2015), arms control agreements (Avenhaus and Kilgour, 2004), and doping in sports (Kirstein, 2014). Each player in this game-theoretic formation has two strategies. The inspector's strategy is to check (or not check) if the inspectee has adhered to a set of rules and the inspectee's strategy is to either break the rules or to comply with them. In this normal-form mixed-strategy game, the driver (the inspectee) parks illegally with probability β and the enforcement unit (the inspector) inspects the driver with a probability α. There are four possible outcomes in this game and the two players obtain a payoff from each outcome. The payoff matrix is presented in Table 1 where the first term in each entry is the driver's payoff and the second term is the enforcement unit's payoff. The payoffs are comprised of the following terms[1]: The driver pays p_0 dollars for parking legally and f dollars for parking illegally and getting cited. The enforcement unit pays c_0 dollars per vehicle inspection. These costs are ideally set up such that $c_0 < p_0 < f$: The condition $c_0 < p_0$ ensures that the enforcement unit has monetary incentive for inspecting the driver and the condition $p_0 < f$ ensures that drivers have incentive to park legally as well illegally. The presented mixed-strategy inspection game has a unique Nash equilibrium with $\beta = c_0/f$ and $\alpha = p_0/f$.

Table 1. Parking enforcement as an inspection game. The two components in each entry are the driver and the enforcement unit payoffs, respectively.

Driver / Enforcement unit	Inspect	Do not inspect
Park Illegally	$(-f, f - c_0)$	$(0,0)$
Park Legally	$(-p_0, p_0 - c_0)$	$(-p_0, p_0)$

There are three major drawbacks in the presented classical game-theoretic approach that hinder its applicability to model parking enforcement. First, the inspection game is not sensitive to the number of enforcement units that are deployed and hence cannot be used to find the optimal level-of-enforcement. Second, the inspection game does not capture how the citation probability is related to the dwell time of illegal vehicles and hence cannot be used to assess illegal parking behavior at a desirable level of detail. Third, the inspection game is limited in replicating realistically how the rate of citations is non-linearly related to the number of illegally parked vehicles and the number of enforcement units in a region (Wright, 1983). To accommodate these features in the inspection game, we use the concept of bilateral searching and matching (or

[1] The enforcement unit payoffs in Table 1 can be revised to represent social welfare instead of profit.

bilateral meeting) which models the search friction between illegal vehicles and enforcement units as the latter searches for the former.

2.3 Bilateral meeting

Bilateral meeting models explicitly quantify the friction between two sets of agents as they seek each other out in an aggregated market. In a frictionless market, the meetings between the two sets of agents would occur instantaneously which is not the case in parking enforcement. Examples of bilateral meeting in economics include taxi-passenger meeting (Yang et al., 2010; Yang and Yang, 2011; Yang et al., 2014), buyer-seller meeting (Burdett et al., 2001), and employer-employee meeting in the labor market (Andolfatto, 1996; Berman, 1997; Barnichon and Figura, 2015). Mathematically, the bilateral meeting process is formulated so that the meeting rate between the two sets of agents (say vacant taxis and passengers) is a function of the size of each set of agents in an aggregated market. As an example, the rate that vacant taxis meet passengers is a function of (i) the number of passengers waiting for a taxi and (ii) the number of vacant taxis cruising to find passengers. This quantification of the meeting rate has time and again proven to be very useful in various topics of economics because it enables the modelling of frictions in otherwise conventional models, with a minimum of added complexity (Petrongolo and Pissarides, 2001). In this paper, we use the bilateral meeting function to model the friction between enforcement units and illegal vehicles. The simplicity of the meeting function allows us to develop an analytical model of parking enforcement which provides insights about the interplay between key factors that influence illegal parking behavior.

The proposed model of parking enforcement is timely for the following reasons. First, the literature on parking is extensive and covers many diverse topics such as pricing of parking facilities (Qian et al, 2012; He et al., 2015; Zheng and Geroliminis, 2016), parking reservation (Yang et al., 2013; Liu et al., 2014), and parking equilibrium (Boyles et al., 2015). However, only a handful of studies are dedicated to parking enforcement despite its practical importance and widespread application in many cities. Second, many studies of transportation systems that are subject to enforcement either disregard the impact of enforcement or assume perfect enforcement (Yang et al., 2012; He et al., 2013). Although these assumptions are valid in many contexts, they can also be limiting in situations where enforcement costs are non-trivial. Third, illegal parking is unique type of violation because it is a time-based. That is, it does not only matter that drivers are parking illegally but how long they engage in the illegal activity (i.e., the dwell time). Hence, some of the findings here can be applied to other similar violations such as breaking the speed limit in freeways where it is important how long a vehicle is driving passed the limit.

3 The choice to park legally or illegally

3.1 Arrival rate, dwell time, and utility of legal and illegal vehicles

Vehicles enter a region at the rate of T [vehicles per hour] and they all need to park. At the top level of the parking choice, each vehicle chooses to park legally or illegally. The arrival rate of illegal vehicles is T^v where v represents a "violator" and the arrival rate of legal vehicles is T^n where "n" represents a "non-violator". The total flow of vehicles is the sum of legal and illegal vehicle flows such that the following equation holds

$$T = T^n + T^v \quad (1)$$

The probability that a randomly arriving vehicle parks illegally is β so that the flow of illegal and legal vehicles is computed as $T^v = T\beta$ and $T^n = T(1-\beta)$, respectively. The probability of parking illegally β depends on the utility received by the driver from parking legally or illegally. Legal vehicles receive a systematic utility of U^n [dollars] and illegal vehicles receive a systematic utility of U^v [dollars]. Hence, assuming that the choice of parking follows a logistic function, the illegal parking probability β can be defined as the following

$$\beta = \frac{\exp(\theta U^v)}{\exp(\theta U^v) + \exp(\theta U^n)} \quad (2)$$

where θ is a non-negative dispersion parameter that can be estimated empirically. The probability β in Eq. (2) is also referred to as the non-compliance ratio (Cullinane and Polak, 1992).

The utilities of parking (U^v and U^n) are calculated as the benefit minus the cost of parking. The benefit of parking is a direct result of the activity that the driver is performing. As an example, individuals gain a benefit from engaging in shopping activities in the downtown core but they first need to find a parking spot to perform the activity. Each driver receives a marginal benefit of $s(l)$ at the l^{th} minute of parking (i.e. parking to perform a given activity). The function $s(l)$ is non-negative, strictly decreasing, convex, and asymptotic to zero, thus implying that travelers always obtain a higher marginal utility from the earlier minutes of parking[2] .

The cost of parking, on the other hand, depends on whether the vehicle is parked legally or illegally and on the duration of parking. Legal vehicles park for l^n hours and pay a variable hourly-based price of p [dollars per hour] so that they incur a total cost of pl^n dollars. Illegal vehicles, on the other hand, park for a period of l^v hours but only have to pay a fine if they are cited by an enforcement unit. The probability that an illegal vehicle is cited by an enforcement unit is denoted by α which is also referred to as the citation probability, the citation fine is denoted by f [dollars], and the expected cost of parking illegally is αf.

With the defined costs and benefits, the two systematic utilities are computed for legal and illegal vehicles, respectively, as

$$U^n = \int_0^{l^n} s(l) \cdot dl \; - pl^n \quad (3)$$

$$U^v = \int_0^{l^v} s(l) \cdot dl - \alpha f \quad (4)$$

3.2 The meeting rate and the citation probability

In this section, we derive the citation probability and identify the factors that influence it. Let N^v be the expected number of illegal vehicles in a region. According to Little's law, N^v is computed as

[2] The properties of the benefit function are consistent with the law of diminishing marginal benefit.

$$N^v = T^v l^v \tag{5}$$

The N^v illegal vehicles are sought out by k enforcement units. The event where an enforcement unit finds an illegal vehicle is called a "meeting"; a meeting is synonymous to citing an illegal vehicle. The meeting rate is denoted by m [citations per hour] and is a function of the number of illegal vehicles N^v and number of enforcement units k as follows

$$m = M(N^v, k) \tag{6}$$

In Eq. (6), we have $\partial m/\partial N^v > 0$ and $\partial m/\partial k > 0$ in their domains $N^v \geq 0$ and $k \geq 0$, which indicates that the meeting rate increases with respect to the number of illegal vehicles or enforcement units. Moreover, $m \to 0$ as either $N^v \to 0$ or $k \to 0$, which indicates that no meetings occur if there are no illegal vehicles or no enforcement units present in the region.

In an aggregate environment, the meeting rate may increase faster or slower than linearly with respect to proportionate increases in the number of enforcement units or the number of illegal vehicles. To present the rate of change in the meeting rate, we define two parameters γ_1 and γ_2 which represent respectively the elasticity of the meeting function with respect to the number of illegal vehicles N^v and the number of enforcement units k at any given time. The two elasticities, within their domains $0 < \gamma_1, \gamma_2 \leq 1$, represent the enforcement technology that is implemented in the region. For instance, a human-based inspection technology is distinguished from a camera-based inspection technology based on the two elasticities; as the inspection technology becomes more efficient, the two elasticities become larger. These elasticities are mathematically defined as

$$\gamma_1 = \frac{\partial M}{\partial N^v} \frac{N^v}{M} \tag{7}$$

$$\gamma_2 = \frac{\partial M}{\partial k} \frac{k}{M} \tag{8}$$

We now use the meeting function *M* to define the citation probability α as the following. According to the law of total expectation, the citation probability α is the ratio of the meeting rate over the arrival rate of illegal vehicles[3] T^v:

$$\alpha = \frac{m}{T^v} \equiv \frac{M(N^v, K^e)}{T^v} \tag{9}$$

Hence, each newly arriving illegal vehicle can get cited with a probability α. To analyze the properties of the citation probability, we take implicit differentiation from both sides of Eq. (9) with respect to l^v (Lemma A1, Appendix A) and derive the following two properties

$$\frac{\partial \alpha}{\partial T^v} = \frac{\alpha \cdot (\gamma_1 - 1)}{T^v} \tag{10}$$

$$\frac{\partial \alpha}{\partial l^v} = \frac{\alpha \gamma_1}{l^v} \tag{11}$$

The following insights are readily obtained from the above properties (Eq. 10 and 11) of the citation probability. First, a higher arrival rate of illegal vehicles T^v eventually lowers the

[3] The citation probability is analogous the probability of finding a job in the job-market literature.

citation probability α because the enforcement units have to search within a larger pool of illegal vehicles; as a result of having more illegal vehicles in the region, each individual illegal vehicle has a lower chance of getting cited. This observation is mathematically confirmed because Eq. (10) is negative since $\alpha, T^v > 0$ and $\gamma_1 \leq 1$. Second, illegal vehicles are more susceptible to getting a citation if they are parked for a long time; the longer a vehicle parks illegally, the higher the chance that an enforcement unit finds the vehicle and cites it. This observation is mathematically confirmed because Eq. (11) is positive since $\alpha, l^v > 0$ and $0 < \gamma_1$. A graphical representation of the two equations is presented in Fig. 2 which illustrates the meeting rate and the citation probability for a given range of illegal arrival rates and dwell times[4]. As illustrated, the meeting rate increases with T^v and l^v whereas the citation probability increases with l^v but decreases with T^v. Third, the two derivatives (Eq. 10 and 11) show that as γ_1 increases from 0 to 1, the citation probability α becomes less sensitive to T^v and more sensitive to l^v. For instance, at $\gamma_1 = 1$, the citation probability is only and highly sensitive to l^v but not sensitive at all to T^v whereas at $\gamma_1 = 0$, the citation probability α depends only on illegal arrival rate T^v. Having defined the citation probability α, we now explain the equilibrium that arises from illegal parking.

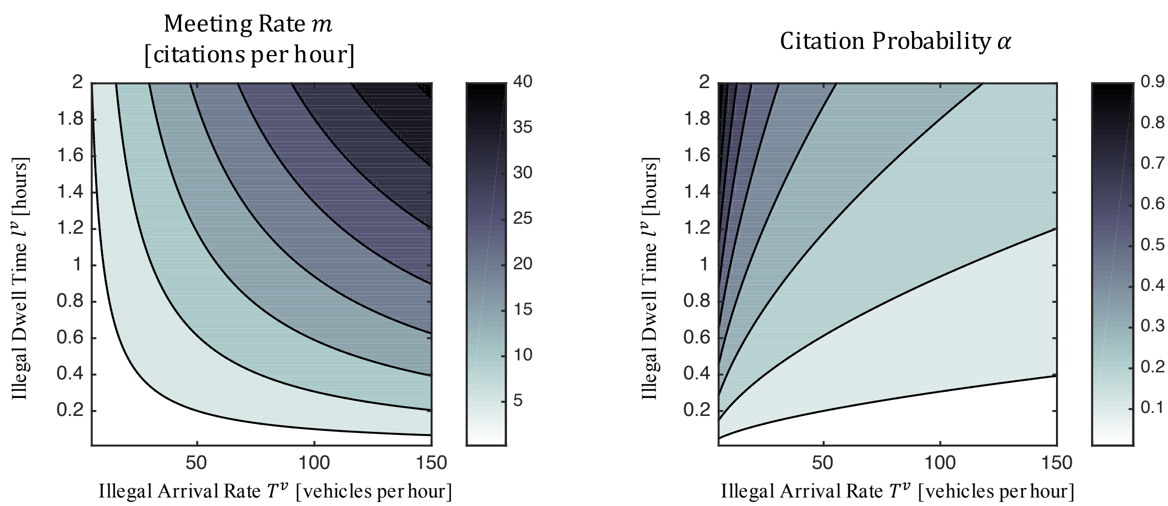

Fig. 2. Meeting rate and citation probability as functions of the illegal arrival rate and well time.

4 An equilibrium model of illegal parking

Let us assume that each driver makes two consecutive decisions when parking. At the upper-level decision, the driver chooses to park legally or illegally and at the lower-level decision, the driver chooses the parking dwell time. This choice structure yields an equilibrium where the equilibrium dwell times (l^{n^*} and l^{v^*}) and the equilibrium arrival rates (T^{n^*} and T^{v^*}) are defined such that the following two conditions are satisfied:

- Condition 1 (lower-level): l^{n^*} and l^{v^*} are chosen to maximize the utilities U^n and U^v.

[4] The meeting rate in Fig. 2 is a Cobb-Douglas function of the form $M = A_0 (N^v)^{\delta_1} (k)^{\delta_2}$ where $A_0 = 0.6, k = 10, \delta_1 = 0.6, \delta_2 = 0.3$.

- Condition 2 (upper-level): T^{n*} and T^{v*} are chosen based on the utilities U^n and U^v obtained from the lower-level.

We now explain each of the two equilibrium conditions in detail, describe their properties, and show how they are intertwined.

4.1 Condition 1 (lower-level): l^n and l^v are chosen to maximize U^n and U^v.

At the lower-level of the equilibrium, the dwell times l^n and l^v are chosen to maximize the utilities U^n and U^v. For legal vehicles, U^n is maximized at l^{n*} based on the first order optimality condition ($dU^n/dl^n = 0$) which leads to the following

$$s(l^{n*}) = p \quad \Leftrightarrow \quad l^{n*} = s^{-1}(p). \tag{12}$$

Eq. (12) shows that l^{n*} and consequently U^n are both only dependent on the parking price *p*. Hence, if the parking price *p* is fixed, then U^n and l^{n*} are both fixed as well. Hereafter, we assume that *p*, l^{n*}, and U^n are all fixed[5].

For illegal vehicles, U^v is maximized at l^{v*} based on the first order optimality condition ($dU^v/dl^v = 0$) which leads to the following

$$s(l^{v*}) = f\frac{d\alpha}{dl^v} \tag{13}$$

To interpret Eq. (13) we first need to compute the full derivative $d\alpha/dl^v$. Given that the citation probability α is a function of l^v and T^v (based on Eq. 10 and 11), the full derivative of $\alpha(l^v, T^v)$ with respect to l^v is:

$$\frac{d\alpha}{dl^v} = \frac{\partial\alpha}{\partial l^v} + \frac{\partial\alpha}{\partial T^v}\frac{dT^v}{dl^v} \tag{14}$$

To simplify Eq. (14), we first show that the second term on the right-side of Eq. (15) is equal to zero at $l^v = l^{v*}$. This can be shown by expanding dT^v/dl^v to

$$\frac{dT^v}{dl^v} = \frac{\partial T^v}{\partial U^v}\cdot\frac{dU^v}{dl^v} \tag{15}$$

and setting $dU^v/dl^v = 0$ at $l^v = l^{v*}$ based on the first order of optimality. Hence, we have $dT^v/dl^v = 0$ in Eq. (15) which simplifies Eq. (14) to

$$\frac{d\alpha}{dl^v} = \frac{\partial\alpha}{\partial l^v} = \frac{\alpha\gamma_1}{l^v} \tag{16}$$

Eq. (16) shows that the partial and full derivatives of α with respect to l^v are equal to each other at l^{v*}. Hence, the citation probability is not sensitive to the arrival rate T^v at l^{v*}. Using the result of Eq. (16), the first order optimality condition in Eq. (13) can now be rewritten as

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[5] The second order optimality condition can also be easily checked to show that U^n is strictly concave and l^{n*} is unique. The legal utility U^n is concave because the benefit function $\int s(l)\cdot dl$ is concave and the cost of legal parking is convex as it increases linearly with the parking dwell time l^n.

$$s(l^{v*})l^{v*} = f\alpha\gamma_1 \quad (17)$$

The first condition of equilibrium is satisfied as long as Eq. (17) is justified. Solving Eq. (17), however, is not particularly easy in its current form because α is a function of both l^v and T^v. To solve Eq. (17), let us define $l^{v*} = L(T^v)$ as a function that finds the optimal illegal dwell time l^{v*} for a given arrival rate T^v. We show in Lemma B2 (Appendix B) that l^{v*} is unique for each T^v. Given the uniqueness of l^{v*}, the challenge now is to find the equilibrium illegal arrival rate T^{v*} that satisfies $l^{v*} = L(T^{v*})$. We explore this condition as the second condition of equilibrium in the next subsection.

The following two remarks can be inferred from Eq. (17). Remark 1 states that in an environment where the optimal dwell times are equal, i.e. $l^{v*} = l^{n*}$, the drivers always receive a higher utility from parking legally. Hence, there is no incentive for any vehicle to park illegally if it obtains a higher utility from parking legally for the same dwell time. An indirect conclusion of this observation is that vehicles are more inclined to park illegally if their dwell time is short and they are more inclined to park legally if their dwell time is long. Remark 1 is presented as the following.

Remark 1: For the same dwell time $l^{v*} = l^{n*}$, legal vehicles receive a higher utility than illegal vehicles such that $U^v < U^n$.

Proof: To present the proof, we need to show that $U^v < U^n$ whenever $l^{v*} = l^{n*} = \hat{l}$. We proceed by investigating the benefit and cost components of the two utilities U^v and U^n. It is easy to establish that the benefit components (first terms in Eq. 3 and Eq. 4) of the two utilities are both equal to each other as they are both $\int_0^{\hat{l}} s(l) \cdot dl$. Hence, it suffices to show that the cost of illegal parking αf is higher than legal parking $p\hat{l}$ such that $U^v < U^n$ holds. Our task now is to prove the following

$$p\hat{l} < \alpha f \quad (18)$$

To show the cost inequality $p\hat{l} < \alpha f$ holds, we use Eq. (12) to rewrite Eq. (17) as $p\hat{l} = f\alpha\gamma_1$. Using this result, Eq. (18) can be rewritten as $\gamma_1 f\alpha < \alpha f$ which is always true when $\gamma_1 < 1$. Hence, Eq. (18) is confirmed and the remark is proven. ■

In Remark 2 we show how γ_1 influences the expected cost of illegal parking αf at the two boundaries $\gamma_1 \to 0$ and $\gamma_1 = 1$. To have $\gamma_1 \to 0$ is analogous to having a fixed parking price and to have $\gamma_1 = 1$ is analogous to having a cost that increases linearly with the dwell time such as in the case of legal parking. The insight from Remark 2 is that the cost of illegal parking is very generic and can have a linear structure similar to the cost of legal parking. Remark 2 is presented as the following.

Remark 2: To have $\gamma_1 \to 0$ is analogous to having a fixed legal parking price and to have $\gamma_1 = 1$ is analogous to having a cost that increases linearly with the dwell time.

Proof: Consider first the case where $\gamma_1 \to 0$. As $\gamma_1 \to 0$, we have, according to Eq. (17), $s(l^{v*})l^{v*} = 0$, which happens when $l^{v*} = 0$ or when $l^{v*} \to \infty$. By showing later that $l^{v*} = 0$ is an unstable solution, we can only have $l^{v*} \to \infty$. The statement $l^{v*} \to \infty$ is analogous to having a fixed parking price so that vehicles pay the fixed price and park as long as they want.

Consider now the case where $\gamma_1 = 1$. When $\gamma_1 = 1$, the citation probability α increases linearly and exclusively with the dwell time l^{v*} which means that the expected illegal cost αf is also only linearly dependent on the dwell time l^{v*}. We first show that α is exclusively dependent on l^{v*} and second show that the dependence is linear as well. Exclusivity is verified as α no longer depends on T^v because $\partial\alpha/\partial T^v = \alpha \cdot (\gamma_1 - 1)/T^v = 0$ at $\gamma_1 = 1$ according to Eq. (10). Linearity is verified as elasticity of α with respect to l^{v*} is strictly equal to 1 because $\partial\alpha/\partial l^v \cdot l^v/\alpha = \gamma_1 = 1$ according to Eq. (11).■

4.2 Condition 2 (upper level): T^n and T^v are dependent on the utilities

At the upper level of the equilibrium, the arrival rates T^n and T^v are materialized according to the obtained utilities U^n and U^v from the lower level of the equilibrium. From the two utilities, the legal utility U^n is assumed to be known and fixed because the legal parking price p is fixed. The illegal utility U^v, however, is itself a function of T^v as any change in T^v influences the citation probability α (Eq. 10) which in turn impacts U^v (Eq. 4). As an example, when T^v increases, the citation probability declines because there are more illegal vehicles to be cited. This decline in the citation probability lowers the expected cost of illegal vehicles $f\alpha$ which in turn compels more drivers to park illegally and consequently increases T^v. Hence, T^v depends on U^v and U^v depends on T^v.

To present this relationship mathematically, let $\Gamma: \mathbf{T^v} \rightarrow \mathbf{T^v}$ be a continuous function mapping a set $\mathbf{T^v}$ of all illegal arrival rates to itself. The function Γ is defined based on Eq. (2) as

$$\Gamma(T^v) = \frac{T}{1+\exp\left(\theta[U^n - U^v(T^v, l^v)]\right)} \qquad \forall\, T^v \in \mathbf{T^v} \tag{19}$$

where $l^v = L(T^v)$. As shown in Eq. (19), we can compute the illegal utility U^v for a given T^v (right-hand-side of Eq. 19) and we can use the computed U^v to recalculate the illegal arrival rate T^v. At equilibrium, the illegal arrival rate T^{v*} must be chosen such that the upper-level equilibrium condition $T^{v*} = \Gamma(T^{v*})$ and the lower-level equilibrium condition $l^{v*} = L(T^{v*})$ are simultaneously justified. The upper level condition ensures that at T^{v*}, no legally parked vehicle likes to change to parking illegally and no illegally parked vehicle likes to change to parking legally. We now explore the properties of this equilibrium and prove that the solution (T^{v*}, l^{v*}) exists and is unique.

5 Properties of the equilibrium

We now analyze the presented equilibrium by proving its existence and uniqueness. We then investigate the properties of the equilibrium and present an algorithm to find the equilibrium solution.

5.1 Existence of an equilibrium

We prove the existence of a steady-state equilibrium that meets the two conditions of an equilibrium by virtue of Brouwer's fixed-point theorem (Fuente, 2000) for the presented non-linear system. Brouwer's fixed point theorem states that if $\Gamma: \mathbf{T^v} \rightarrow \mathbf{T^v}$ is a continuous function mapping a compact and convex set $\mathbf{T^v}$ to itself, then there exists a $T^v \in \mathbf{T^v}$ such that $\Gamma(T^v) = T^v$. To prove the existence of an equilibrium, first we show that the set $\mathbf{T^v}$ is a compact and convex set (Lemma B1, Appendix B), and second we establish that continuity condition is satisfied (Lemma B3, Appendix B). Using Lemma B1 and Lemma B3, we readily prove the existence of an equilibrium. The proof and the two lemmas are presented in Appendix B.

5.2 Uniqueness of the equilibrium

We prove that the equilibrium (l^{v^*}, T^{v^*}) is unique by showing that it occurs at the intersection of the two (blue) curves L and Γ in Fig. 3. The proof consists of first defining the two curves L and Γ and showing that they cross each other at a maximum of two points, one of which is the equilibrium. As a prerequisite to the proof, consider the two-dimensional space illustrated in Fig. 3 with l^v on the horizontal axis and T^v on the vertical axis. In this space, two $N^v (\equiv T^v l^v)$ contours are plotted (in red) for N_1^v and N_2^v such that $N_1^v < N_2^v$.

The two curves L and Γ are defined as follows with slight abuse of notation of the function L. The first curve $L(N^v)$ represents the dwell time of a newly arriving illegal vehicle when N^v vehicles are already parked illegally. This curve is obtained by simply dividing the two sides of Eq. (17) (first equilibrium condition) by l^{v^*} which leads to

$$s(l^{v^*}) = mf\gamma_1 / N^v \tag{20}$$

We precisely define $L(N^v)$ as the solution of Eq. (20) which is strictly a function N^v because *m* is also a function of N^v when all other parameters are fixed.

The second curve $\hat{T}^v = \Gamma(L(N^v))$ is defined as follows. When the illegal dwell time is fixed at $L(N^v)$, the illegal utility has a value that attracts an illegal arrival rate of $\hat{T}^v$. Let us further explain this concept graphically. Consider the contour N_1^v in Fig. 3 where $l_1^v = L(N_1^v)$ is the optimal dwell time occurring at the arrival rate $T_1^v = N_1^v / l_1^v$. For the given dwell time l_1^v and arrival rate T_1^v, the arrival rate should ideally rise from T_1^v to $\hat{T}_1^v$ as more vehicles are inclined to park illegally because of the high utility obtained from the parking dwell time l_1^v. Loosely speaking, $\hat{T}_1^v$ is the induced illegal arrival rate when the dwell time is fixed is l_1^v. It is clear that the equilibrium occurs at the point where the two curves cross. This crossing point (l^{v^*}, T^{v^*}) occurs when the induced demand $\hat{T}^v$ is equal to the materialized demand T^{v^*} (see proof of existence) as is presented in Fig. 3. At this equilibrium solution, the illegal dwell time $l^{v^*} = L(N^{v^*})$ is defined such that the induced demand

Given the definitions of L and Γ, we continue the proof by showing that the equilibrium (l^{v^*}, T^{v^*}) is unique if the crossing point is unique. To show that the crossing point is unique, we prove that there are two points where the two curves cross and only one of the two points is the stable equilibrium. To show that there is a maximum of two crossing points, it is sufficient to show the following three properties hold: (i) both curves L and Γ are increasing (with respect to l^v), (ii) the curve L is convex, and (iii) the curve Γ is concave. We prove in Lemma 1, below, that the first property holds for L but do not present the proofs of the other properties as they are tedious and less interesting. We now show that from the two crossing points, only one of them is the equilibrium by showing that the non-equilibrium point occurring at $T^v = 0$ and $l^v = 0$ is not stable. The instability of this point is shown as follows. By letting $T^v = 0$, we have a very low citation probability which encourages illegal vehicles to park for a very long time such that $l^v \rightarrow \infty$. This long dwell time pushes the solutions away from the original crossing point of $T^v = 0$, $l^v = 0$ and shows the instability of the first solution. Hence, the equilibrium is unique.

Lemma 1. The curve $L(N^v)$ is strictly increasing with l^v.

Proof: Consider $l_1^v = L(N_1^v)$ and $l_2^v = L(N_2^v)$ as shown in Fig. 3. We show here that moving from N_1^v to N_2^v strictly increases the function $L(N^v)$. The proof is derived from Eq. (20) and can be shown in two steps. First, as N^v increases, both the denominator (because of N^v) and numerator (because of *M*) of Eq. (20) increase. However, the numerator increases at lower pace because $\partial M/\partial N^v \cdot N^v/M = \gamma_1$ which is smaller than 1. This shows that increasing N^v reduces $s(l^{v*})$ which is left-hand-side of Eq. (20). The second step is to show that the reduction in $s(l^{v*})$ is equivalent to increasing l^{v*}. This is a straightforward result of the property of the function *s* which is a strictly decreasing function of l^v. Thus, the two steps of the proof show that increasing N^v leads to a higher l^{v*}. ■

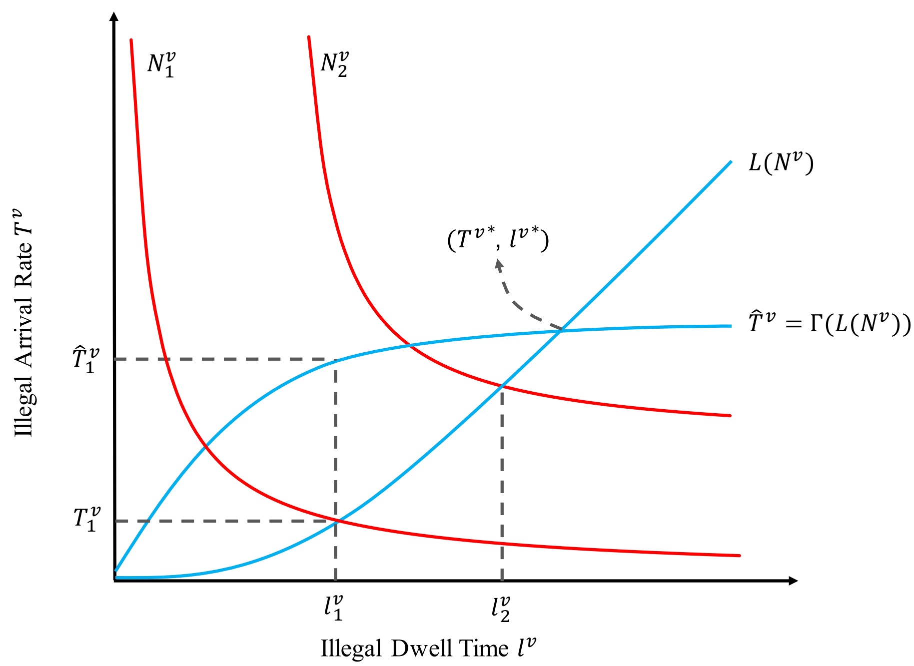

Fig. 3. Equilibrium solution.

5.3 Algorithm

We present an algorithm to find the equilibrium (l^{v*}, T^{v*}). Before explaining the steps of the algorithm, we use the example in Fig. 4 to illustrate graphically how the algorithm searches the solution space to find the equilibrium. We start off with N_0^v as an initial number of illegal vehicles which is chosen randomly. Given N_0^v, we find the optimal dwell time $l_0^v = L(N_0^v)$ and the corresponding arrival rate $T_0^v = N_0^v/l_0^v$. The point (T_0^v, l_0^v) is the starting point of the algorithm. Next, we find the induced demand T_1^v which is larger than T_0^v thus showing that illegal parking is still profitable for drivers. As the illegal arrival rate increases, the number of illegal vehicles also increases from N_0^v to N_1^v. Now, we repeat the procedure by finding $l_1^v = L(N_1^v)$ and continuing the process until we reach the equilibrium. The steps of the algorithm are presented as follows.

Step 1- Initialization

Set the counter $a := 0$. Set the number of illegal vehicles N_a^v to a randomly selected value.

Step 2- Illegal dwell Time $L(N_a^v)$

Use the Newton-Raphson algorithm to solve Eq. (17) and find the optimal illegal dwell time $L(N_a^v)$.

Step 3- Find the materialized demand $T_a^v = N_a^v / L(N_a^v)$.

Step 4- Find the induced demand $\hat{T}^v = \Gamma(\hat{T}^v, L(N_a^v))$ using the following sub-steps.

Step 4.1- Set the counter $b := 0$. Choose a random illegal arrival rate $\hat{T}_b^v$.

Step 4.2- Find the illegal utility U^v as a function of $\hat{T}_b^v$ and $L(N_a^v)$ using Eq. 4.

Step 4.3- Set $b := b + 1$ and find $\hat{T}_b^v = \Gamma(\hat{T}_{b-1}^v, L(N_a^v))$ using Eq. 19.

Step 4.4- Inner convergence check: Go to Step 5 if the following convergence condition is satisfied. Otherwise, go to step 4.2. The convergence condition is the following

$$\left|\hat{T}_b^v - \hat{T}_{b-1}^v\right| \leq \varepsilon \tag{21}$$

Step 5- Update the optimal illegal arrival rate and number of illegal vehicles.

Set $a := a + 1$. Let $T_a^v = \hat{T}_b^v$ and let $N_a^v = T_a^v \cdot L(N_{a-1}^v)$.

Step 6- Outer convergence check

Terminate the algorithm if the following convergence condition is satisfied. Otherwise, go to Step 2. The convergence condition is the following

$$|T_a^v - T_{a-1}^v| \leq \varepsilon \tag{22}$$

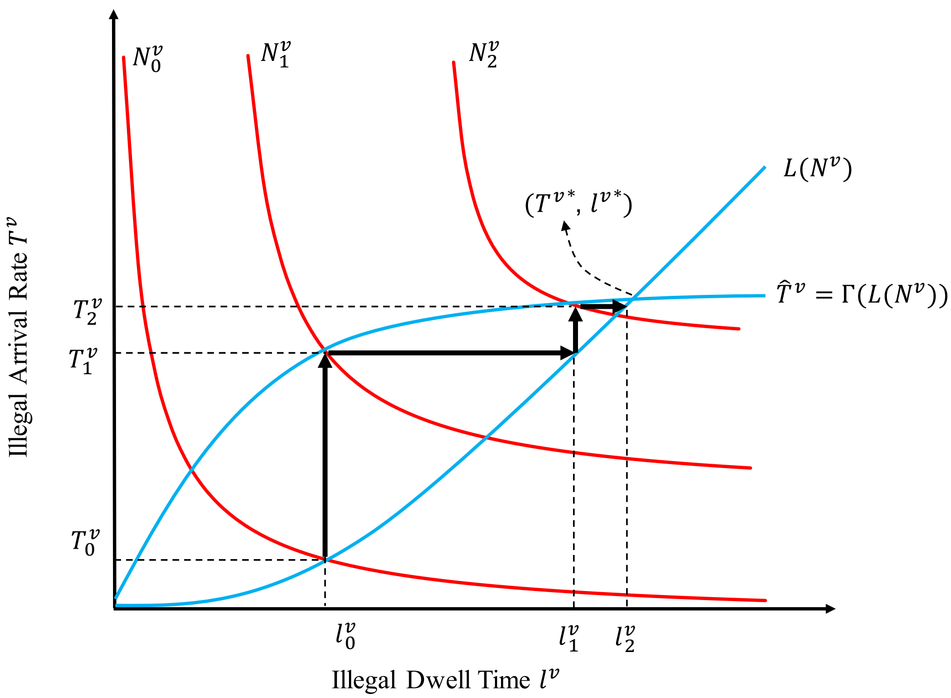

Fig. 4. Steps of the algorithm.

5.4 Elasticities at the equilibrium solution

We now present the elasticity of key variables with respect to *f* and *k* at the equilibrium solution (l^{v*}, T^{v*}). Defining these elasticities improves substantially the process of finding the optimal enforcement policy which involves choosing the appropriate values of *f* and *k*. To denote any elasticity, we use the notation μ_x^y to show that a one percent increase of $x = \{f, k\}$ will change $y = \{m, T^v, l^v\}$ by μ_x^y percent. Although some of the derived the elasticities are not easily interpretable in their current form, we show in the next section that they can be simplified under special (deterministic and random) equilibrium conditions where they provide meaningful insight. Moreover, the defined elasticities are helpful in finding the optimal enforcement policies in the next section. The derivation of all elasticities is presented in Appendix C.

We start off with the elasticity of the meeting rate *m* with respect to the citation fine *f* and level-of-enforcement *k*. As shown in Eq. (23), μ_f^m depends on the three elasticities γ_1, $\mu_f^{l^v}$, and $\mu_f^{T^v}$; the first elasticity γ_1 shows that the impact of the citation fine *f* on the meeting rate *m* is contingent on what kind of inspection technology is implemented and the second two elasticities $\mu_f^{T^v}$ and $\mu_f^{l^v}$ show that the number of illegal vehicles N^v is also influential. Eq. (24) shows that the elasticity μ_k^m is linearly dependent on μ_f^m.

$$\mu_f^m = \gamma_1(\mu_f^{l^v} + \mu_f^{T^v}) \tag{23}$$

$$\mu_k^m = \gamma_2 \mu_f^m \tag{24}$$

We now proceed to define the elasticity of the illegal arrival rate T^v and dwell time l^v with respect to *f* and *k*. These elasticities are presented in Eq. (25)-Eq. (28).

$$\mu_f^{T^v} = \frac{-(1-\beta)\theta f\alpha}{1+(1-\beta)(1-\gamma_1)\theta f\alpha} \tag{25}$$

$$\mu_f^{l^v} = \frac{s(l^v)/[s'(l^v)l^v + s(l^v)(1-\gamma_1)]}{1+(1-\beta)(1-\gamma_1)\theta f\alpha} \tag{26}$$

$$\mu_k^{T^v} = \gamma_2 \mu_f^{T^v} \tag{27}$$

$$\mu_k^{l^v} = \gamma_2 \mu_f^{l^v} \tag{28}$$

6 Optimal parking enforcement policies

The two commonly pursued objectives of parking enforcement are profit maximization and social welfare maximization. To reach either of the two objectives, parking authorities implement strategic policies by choosing the citation fine f and the level-of-enforcement k because of their explicit influence on the parking behavior. The effect of a policy, comprised of the pair (k, f), on each objective is investigated in this section.

6.1 Maximizing the profit of parking enforcement

The profit of parking enforcement is the revenue generated from the tickets that are issued minus the cost of employing enforcement units. The expected revenue is the defined as fm [dollars per hour] which is the product of the citation fine and the number of citations in one hour. To define the cost of enforcing parking, let c be fixed the cost of acquiring one enforcement unit for one hour so that the total cost of enforcement is ck [dollars per hour]. Then, the expected profit from parking enforcement is denoted by π and calculated as

$$\pi = fm - ck \tag{29}$$

To maximize the profit π, we take the derivative of Eq. (29) as

$$\frac{d\pi}{df} = f\frac{dm}{df} + m \equiv m\left(1 + \mu_f^m\right) \tag{30}$$

expected profit is maximized by setting $d\pi/df = 0$ or equivalently

$$\mu_f^m = -1 \tag{31}$$

which implies that at the optimal citation fine f^*, a one percent increase in the fine must decrease the meeting rate by one percent. This decline in the meeting rate occurs with either a decrease the arrival rate T^v or the dwell time l^v or both. Hence, at f^* the number of illegal vehicles N^v ($\equiv T^v \cdot l^v$) is negatively impacted with an increase in the fine. We later explore, in the next section, how f^* can be calculated under special deterministic and random equilibrium conditions.

The second influencing factor in the profit π is the number of enforcement units k. To find the optimal k^*, we take the derivative of Eq. (29) as

$$\frac{d\pi}{dk} = \frac{fm}{k}(\gamma_2 + \mu_k^m) - c \tag{32}$$

By setting $d\pi/dk = 0$, we have

$$\mu_k^m = \frac{ck}{mf} - \gamma_2 \tag{33}$$

which shows that the optimal elasticity μ_k^m is equal to the cost-benefit ratio ck/mf offset by the second technology parameter γ_2. Eq. (33) yields the following two insights. When $\frac{ck}{mf} < \gamma_2$, we have $\mu_k^m < 0$ which indicates that the inspection technology is efficient enough (because of a high γ_2 value) to the point where increasing the level-of-enforcement lowers the meeting rate as vehicles stop parking illegally to avoid getting cited. On the other hand, when $\frac{ck}{mf} > \gamma_2$, we have $\mu_k^m > 0$ which shows that the lack of efficient inspection technology (because of a low γ_2 value) must be rectified by deploying enough enforcement units to cite the illegal vehicles.

Let us now consider the case of optimizing simultaneously the pair (k^*, f^*) to maximize the profit π. Given that $\mu_k^m = \gamma_2 \mu_f^m$ (Eq. 24), we can rewrite Eq. (32) as

$$\frac{d\pi}{dk} = \left(\frac{f\gamma_2}{k} \cdot \frac{d\pi}{df}\right) - c \tag{34}$$

which shows that the profit π cannot be simultaneously maximized with respect to *f* and *k*; that is the two terms $d\pi/f$ and $d\pi/dk$ cannot be equal to zero together. By setting $d\pi/df = 0$, we have $d\pi/dk = -c$ which implies that lowering the level-of-enforcement at f^* can further raise the generated profit. This indicates that, ideally, it is most profitable to have a very small level-of-enforcement and a high citation fine so that a few illegal vehicles are caught with little inspection but they pay a large sum of money. Practically, however, this type of policy cannot be applied as there are social restrictions that limit the upper bound of the citation fine. Hence, in real-life scenarios, the optimal citation fine must be set to the largest socially acceptable price. When the fine is set, the level-of-enforcement can be obtained using Eq. (32).

6.2 Profit maximization under stochastic equilibrium conditions

To further understand the underlying factors that influence the expected profit, we investigate two special cases of the stochastic equilibrium. The first case is to have a complete random choice between parking legally or illegally. This choice is analogous to setting the dispersion parameter of the logit choice model (Eq. 2) to $\theta = 0$. The second case is to have a complete deterministic choice which is equivalent to having a dispersion parameter of $\theta \to \infty$.

Random choice of parking ($\theta = 0$)

Under the random choice structure, the drivers randomly choose to park legally or illegally as they are indifferent between the two options. Hence, 50% of the vehicles park legally and the other 50% park illegally which leads to $T^v = T^n = 0.5T$. Consider now the optimality condition $\mu_f^m = -1$ (Eq. 31) for maximizing profit with respect to *f*. This optimality condition can be rewritten (according to Eq. 23) as the following

$$\gamma_1\left(\mu_f^{T^v} + \mu_f^{l^v}\right) = -1 \tag{35}$$

We now analyze the elasticities $\mu_f^{T^v}$ and $\mu_f^{l^v}$. By setting $\theta = 0$, it is evident that $\mu_f^{T^v} = 0$ because the drivers are making a random decision to park and their decision is not influenced by the citation fine[6]. Hence, the optimality condition in Eq. (35) with $\mu_f^{T^v} = 0$ becomes $\gamma_1 \mu_f^{l^v} = -1$ which can be simplified (Lemma D2, Appendix D) to

$$l^v = -1/\delta \tag{36}$$

where δ is a fixed parameter. Eq. (36) implies the following two conclusions at $\theta = 0$:

- First, the illegal dwell time that provides the maximum profit depends only on δ which is a fixed parameter. Hence, the optimal f^* must be chosen such that the equilibrium dwell time l^{v*} reaches the value $l^{v*} = -1/\delta$. Otherwise, if $f > f^*$ then $l^v < l^{v*}$ and the illegal vehicles are not parked for a long enough time to get cited. Alternatively, if $f < f^*$ then vehicles receive a citation but they do not pay a substantial penalty for it. Under both cases, the parking authorities lose profit unless the optimal citation fine f^* is selected[7].
- The second conclusion is that the optimal citation fine f^* is inversely related to efficiency of the inspection technology γ_1; as γ_1 decreases f^* must increase to make up for the inefficient inspection technology. This conclusion can be tested mathematically using the first condition of equilibrium (Eq. 17): given that $l^{v*}$and $T^{v*}$are both fixed at f^*, the product $f\gamma_1$ becomes a fixed value which shows that f and γ_1 are inversely related.

Deterministic choice of parking ($\theta \to \infty$)

Consider now the case of the deterministic choice where $\theta \to \infty$. In this choice structure, the drivers are completely aware of the environment and choose the best option of parking that maximizes their obtained systematic utility. In such an environment, Wardrop's principle states that no traveler can change his/her parking choice (the choice of parking legally or illegally) and reach a higher net utility. This equilibrium emerges when, for $T^v, T^n > 0$, we have $U^n = U^v$. Hence, for the case of the deterministic choice, the two equilibrium conditions of Section 4 can be redefined as:

$$s(l^{v*})l^{v*} = f\alpha\gamma_1 \tag{37}$$

$$U^n = U^v \quad \Leftrightarrow \quad s(l^{v*}) - f\alpha = s(l^{n*}) - pl^{n*} \tag{38}$$

where Eq. (37) is the first condition (lower level) and Eq. (8) is the second condition (upper level) of the equilibrium. A simple division of the two equilibrium conditions leads to

$$\frac{s\prime(l^{v*})l^{v*}}{s(l^{v*}) - U^n} = \gamma_1 \tag{39}$$

which shows that the equilibrium dwell time l^{v*} is independent of the citation fine f (because the citation fine does not appear in the Eq. 39) and depends only on the two fixed values of γ_1 and U^n.

Consider now the optimally condition $\gamma_1\left(\mu_f^{T^v} + \mu_f^{l^v}\right) = -1$ (Eq. 35). Because the optimality condition depends on the two elasticities $\mu_f^{T^v}$ and $\mu_f^{l^v}$, we investigate individually each of the two

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[6] The condition $\mu_f^{T^v} = 0$ can also be checked by setting $\theta = 0$ in Eq. (25).

[7] The actual value of the citation fine is $f^* = -\frac{s(-1/\delta)}{\gamma_1 \delta(0.5T, -1/\delta)}$ which is proved in Appendix D.

elasticities. For the first elasticity $\mu_f^{l^v}$, as l^{v*} is independent of f (see Eq. 39), the respective elasticity is equal to zero, i.e., $\mu_f^{l^v} = 0$. For the second elasticity, $\mu_f^{T^v}$, we have:

$$\mu_f^{T^v} = \lim_{\theta \to \infty} \frac{-(1-\beta)\theta f \alpha}{1+(1-\beta)(1-\gamma_1)\theta f \alpha} = \frac{1}{(1-\gamma_1)} \tag{40}$$

The relationship $\mu_f^{T^v} = \frac{1}{(1-\gamma_1)}$ in Eq. (40) provides the following two conclusions when at $\theta \to \infty$.

- First, as $0 < \gamma_1 \leq 1$, we have $\mu_f^{T^v} > 1$, which implies that increasing the citation fine f will actually increase the arrival rate T^v. Although this is a non-intuitive result, it has a logical reason. If all drivers are completely aware of the environment, they can collaboratively park illegally and thus reduce the citation probability and hence the expected cost of getting cited for all of them. Mathematically, given that αf is fixed (according to Eq. 38 and Eq. 39), an increase in f should be accompanied with a decrease in $\alpha(l^v, T^v)$. However, given that l^{v*} is also fixed, α will only decrease as a result of increasing the arrival rate T^{v*}. Hence, a higher citation fine leads to a higher arrival rate T^{v*}. In real-life scenarios, this case could only likely occur in mass events (such as major sporting events) where drivers park illegally because they speculate the that everyone else is doing the same.
- The second conclusion of Eq. (40) is that the citation fine must be very large for the deterministic equilibrium. Given that $\mu_f^{T^v} > 1$, the optimality condition $\gamma_1 \mu_f^{T^v} = -1$ is never justified and $\frac{d\pi}{df} > 0$. Hence, under the deterministic equilibrium, the parking authorizes should increase the fine as much as possible to gain a larger expected revenue.

6.3 Maximizing social welfare

In defining social welfare for parking, both legal and illegal drivers must be taken into account. Illegal vehicles add a negative externality to the system since they increase congestion by creating bottlenecks in the traffic flow as a result of parking in appropriate places. Legal vehicles, on the hand, are also important as they have not violated the parking laws and therefore should be rewarded for their behavior. Mathematically, social welfare in the context of parking enforcement is defined as follows. Let $W(f, k)$ denote the social welfare as a function of the citation fine f and the level-of-enforcement k as

$$W(f, k) = T^n \int_0^{l^n} s(l) \cdot dl - Q(N^v) - cN^v \tag{41}$$

where the first term is the net benefit obtained by all legal vehicles, the second term is a function Q which defines the negative externality caused by all illegal vehicles N^v, and the third term is the total cost of enforcement. Arguably, one could also consider the total benefit of the illegal vehicles $T^v \int_0^{l^v} s(w)dw$. This consideration, however, is quite controversial as enforcement policies are implemented to reduce illegal parking when social welfare is of interest.

By taking the derivative of W with respect to f and k, we have the following two equations

$$\frac{dW}{df} = -\frac{T^v}{f}\left[\mu_f^{T^v}\int_0^{l^n} s(w)dw - l^v g'(N^v)\left[\mu_f^{T^v} + \mu_f^{l^v}\right]\right] \tag{42}$$

$$\frac{dW}{dk} = -\frac{T^v\gamma_2}{k}\left[\mu_f^{T^v}\int_0^{l^n} s(w)dw - l^v g'(N^v)\left[\mu_f^{T^v} + \mu_f^{l^v}\right]\right] - c \tag{43}$$

Eq. (42) and (43) cannot be easily investigated analytically. Hence, similar to the previous section on profit maximization, we investigate deterministic and random cases of the stochastic equilibrium.

6.4 Social welfare maximization under stochastic equilibrium conditions

Random choice of parking ($\theta = 0$)

We first present the changes in social welfare with respect to the fine *f*. Under the random choice, as shown previously, we have $\mu_f^{T^v} = 0$ and $\mu_f^{l^v} = \frac{1}{\delta l^v + 1 - \gamma_1}$ (Lemma D1, Appendix D). These elasticities simplify Eq. (42) to

$$\frac{dW}{df} = -\frac{l^v T^v \mu_f^{l^v} g\prime}{f} \tag{44}$$

Eq. (44) is still not easily interpretable because it has the term $\mu_f^{l^v}$. Let us further simplify Eq. (44) by considering the two boundaries $\gamma_1 \to 0$ and $\gamma_1 \to 1$ of the inspection technology. Both boundaries lead to the following relationship

$$\frac{dW}{df} = -\frac{T^v g\prime}{f\delta} \tag{45}$$

which provides the following insights

- Social welfare is progressively improved as the citation fine *f* is increased because $dW/df > 0$ in Eq. (45) since $g' < 0$. This conclusion is also valid in real-life cases as increasing *f* is an easy and straightforward method of deterring illegal parking behavior.
- Increasing the citation fine has a more substantial impact on social welfare when the total marginal externality $T^v g'$ is large. In other words, if a lot of vehicles park illegally and each vehicle creates a lot of delay, then the citation *f* becomes critical in eliminating such behavior. On the other, if $T^v g' \approx 0$, such as in suburban areas where traffic is low, then increasing the fine does not substantially improve social welfare.

We now present the changes in social welfare with respect to the level-of-enforcement k. Using the two elasticities $\mu_f^{T^v}$ and $\mu_f^{l^v}$, the derivative dW/dk in Eq. (43) can be simplified to

$$\frac{dW}{dk} = -\frac{T^v l^v \gamma_2 g'}{k(\delta l^v + 1 - \gamma_1)} - c \tag{46}$$

As Eq. (46) is still not easy to interpret, we further simplify it by considering the two boundaries $\gamma_1 \to 0$ and $\gamma_1 \to 1$ of the inspection technology. Both boundaries lead to the following relationship which uses the condition $dW/dk = 0$ to compute the optimal level-of-enforcement k^* as

$$k^* = -\frac{\gamma_2 T^v g\prime}{c\delta} \tag{47}$$

Eq. (47) provides the following insights:

- The optimal k^* decreases with the cost *c* of each enforcement unit which is a logical and intuitive result.
- The optimal k^* decreases with the technology parameter γ_2; as $\gamma_2 \to 0$ there should no longer be any parking enforcement because the enforcement units cannot find the illegal vehicles.
- The optimal k^* also depends on the total marginal externality $T^v g'$ as was the case in Eq. (45). Enforcement should be intensified when the total marginal externality is high.

7 Numerical experiment

In this section we provide a numerical experiment to illustrate our findings. The three main inputs in the analyses are (i) the marginal benefit function s, (ii) the meeting function M, and (iii) the logit choice model dispersion parameter θ. These inputs are defined as follows. The marginal benefit function has the following form

$$s(l) = B_0 . (B_1)^l \tag{48}$$

where $B_0 = 30$ and $B_1 = 0.2$ are fixed parameters. The function $s(l)$ in Eq. (48) has all the required features of the marginal benefit function because it is strictly decreasing, convex, and asymptotic to zero. The second input is the meeting function which is defined using the Cobb-Douglas relationship as the following

$$M(N^v, k) = A_0 (N^v)^{\gamma_1} (k)^{\gamma_2} \tag{49}$$

where $A_0 = 1$ is a parameter and γ_1, γ_2 are the elasticities. We perform sensitivity analysis on the elasticities to analyze numerically their impact on the equilibrium and the optimal enforcement policy. The third input is the dispersion parameter which is set to $\theta = 0.1$ unless stated otherwise. Sensitivity analysis is performed on θ as well. Finally, the cost of legal parking is set to *p* = 3 [dollars per hour] which leads to a legal parking duration of $l^{n*} = 0.68$ [hours].

7.1 Analysis of the optimal policy

We first assess the impact of each policy, made of the pair (k, f), on the equilibrium. The results are presented in Fig. 5 where the two-dimensional space with *k* on the horizontal axis and *f* on the vertical axis is used to illustrate changes in the illegal arrival rate T^{v*}, dwell time l^{v*}, meeting rate *m*, and citation probability α. The following insights are observed in Fig. 5. First, the illegal arrival rate and dwell time are shown to be highest when *f* and *k* are both low because (i) illegal vehicles have a low chance of getting cited and (ii) even if they get cited, they pay only a small penalty; increasing either *f* or *k*, however, lowers both T^{v*} and l^{v*}. Second, the meeting rate *m* and the citation probability α are shown to increase with *k* and decrease with *f* for the following reasons. The increase with *k* occurs because more enforcement units can catch more illegal vehicles. The decrease with *f*, on the other hand, occurs because fewer vehicles park illegally when *f* is large to avoid a large penalty which leads to a lower meeting rate. Third, the white region in Fig. 5 illustrates policies where *f* and *k* are so large that no vehicle parks illegally. We call this the "no illegal parking" region.

Next, we asses the impact of each policy, i.e. pair (k, f), on the profit π and social welfare W. The results are presented in Fig. 6 and the following insights are observed. First, the maximum profit occurs at a low k and a large f which shows that it is best to have a few enforcement units that cite illegal vehicles for a large penalty. Second, the profit in the “no illegal parking” region (white area in Fig. 5) is negative because the city does not make any citation money is this zone regardless of the level-of-enforcement. Third, the optimal social welfare occurs in the “no illegal parking” region as no negative externality, associated with illegal parking, is imposed. Moreover, the optimal social welfare occurs at a k to avoid the cost deploying the enforcement units. Instead, vehicles are deterred to park illegally due to the high cost of the fine f.

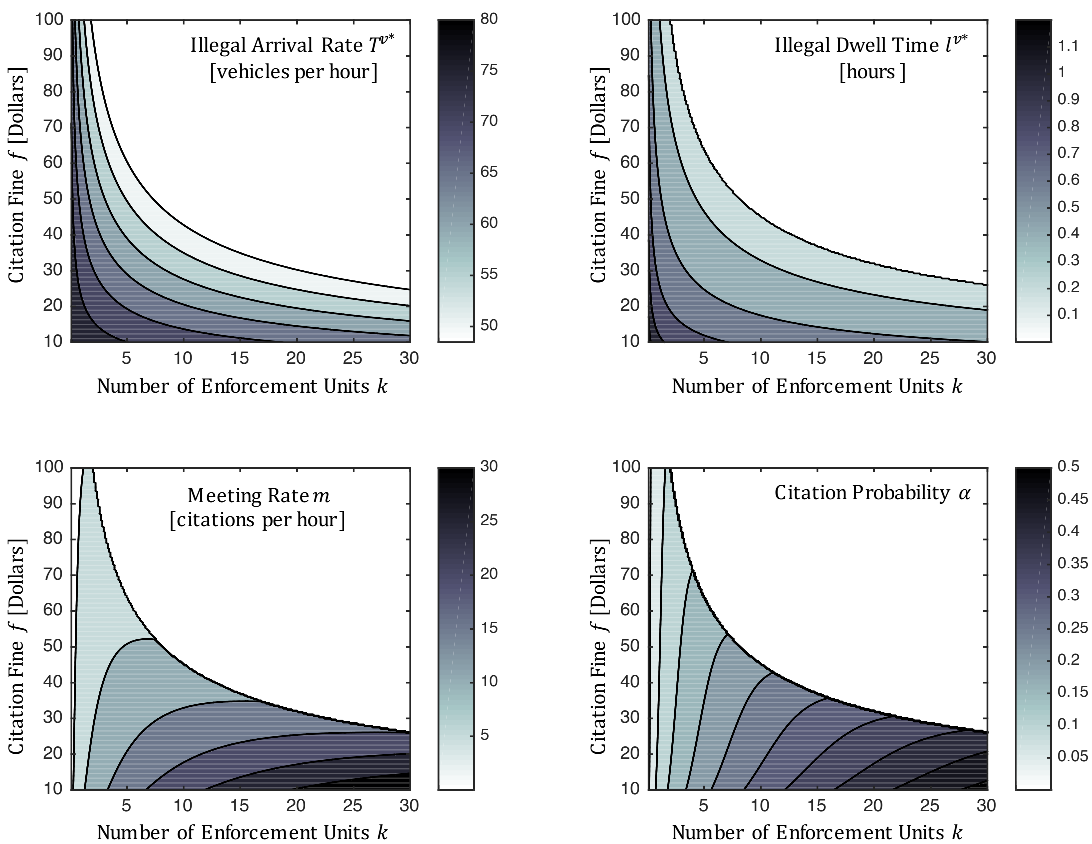

Fig. 5. Impact of each enforcement policy on the illegal arrival rate, dwell time, meeting rate, and citation probability.

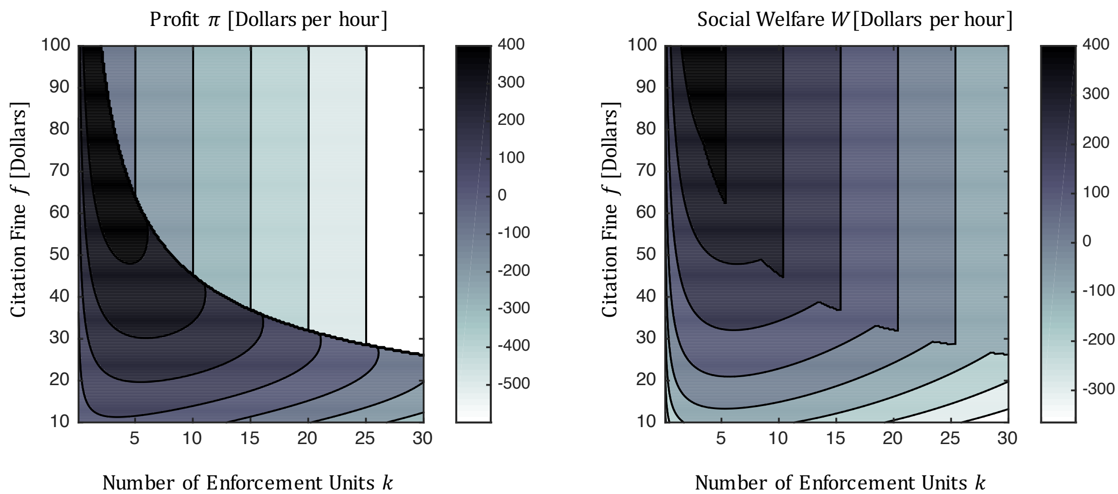

Fig. 6. Profit and social welfare at different enforcement policies.

7.2 Analysis of meeting technology

We investigate the impact of the meeting technology parameters γ_1 and γ_2 on profit and social welfare. We present the impact of simultaneously changing γ_1 and f on the profit. The results are presented in Fig. 7 and the following are observed. First, the lowest profit is obtained at highest γ_1 because vehicles avoid parking illegally due to the efficient inspection technology. We show in the next section that the impact of γ_1 on profit is substantially dependent on the dispersion parameter θ. Second, the citation probability α, illegal arrival rate T^{v*}, and dwell time l^{v*} all decrease with the fine f. Their rate of change, however, depends on the elasticity γ_1; a larger γ_1 increases the rate of change in α, T^{v*}, and l^{v*}. This observation shows the parking behavior is more sensitive to the enforcement policy when the inspection technology is associated with a large γ_1. Third, the two elasticities $\mu_f^{l^v}$ and $\mu_f^{T^v}$ are both negative for all values of citation fine, thus confirming that both T^{v*} and l^{v*} decrease with f.

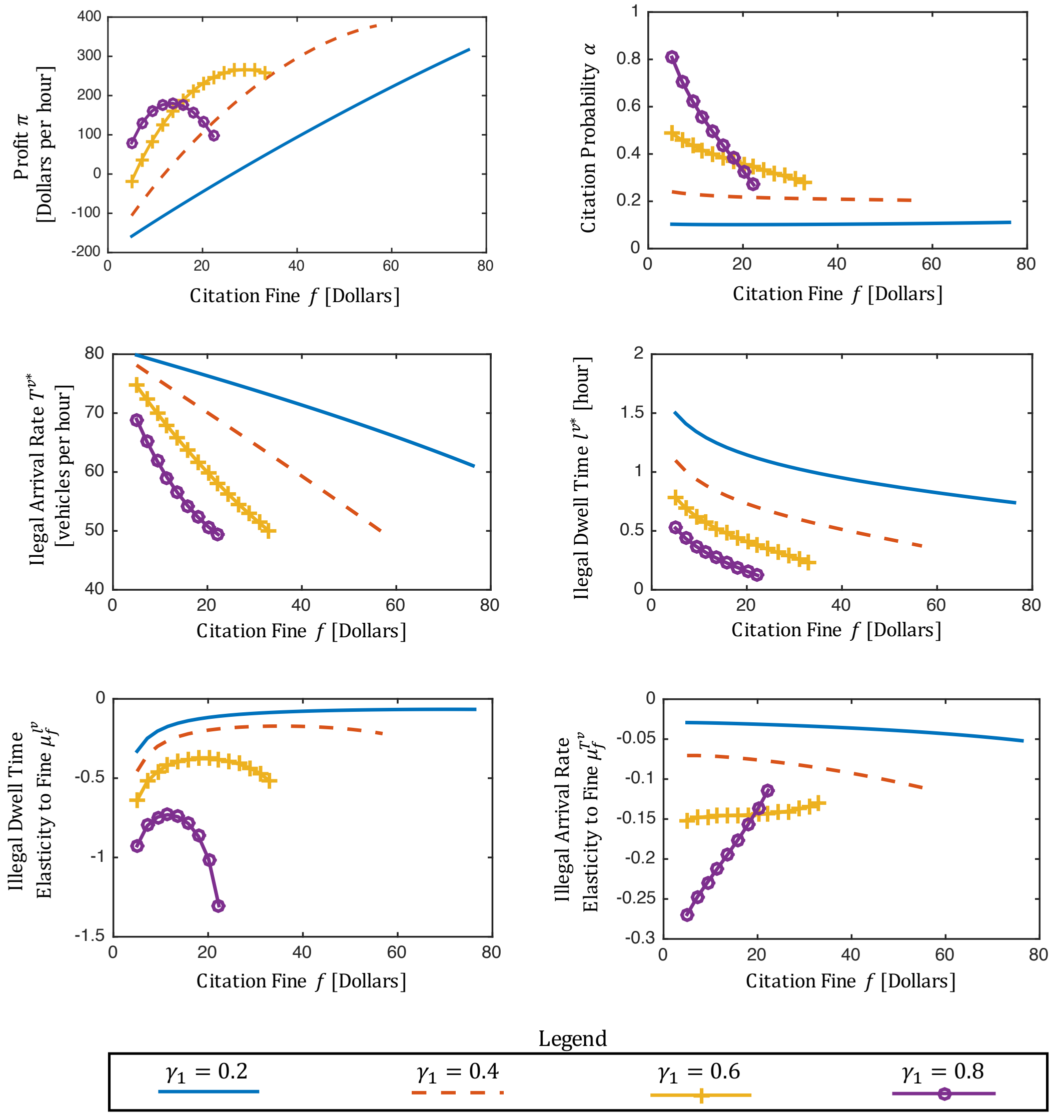

Fig. 7. Impact of the meeting elasticity γ_1 and citation fine f on profit π.

Next, we present the impact of changing γ_2 and k on social welfare. The results are presented in Fig. 8 and the following are observed. First, social welfare is negative when k is too low because too many vehicles park illegally. Similarly, social welfare is also negative when k is too large because of the high cost of hiring the enforcement units. Second, citation probability α is shown to increase with k because more enforcement units are searching for illegal vehicles. Third, both illegal arrival rate T^{v^*} and dwell time $l^{v^*}$are shown to decrease with k because of the higher citation probability α. Fourth, the two elasticities $\mu_f^{l^v}$ and $\mu_f^{T^v}$ are both negative for all values of k, thus confirming that both T^{v^*} and l^{v^*} decrease with k.

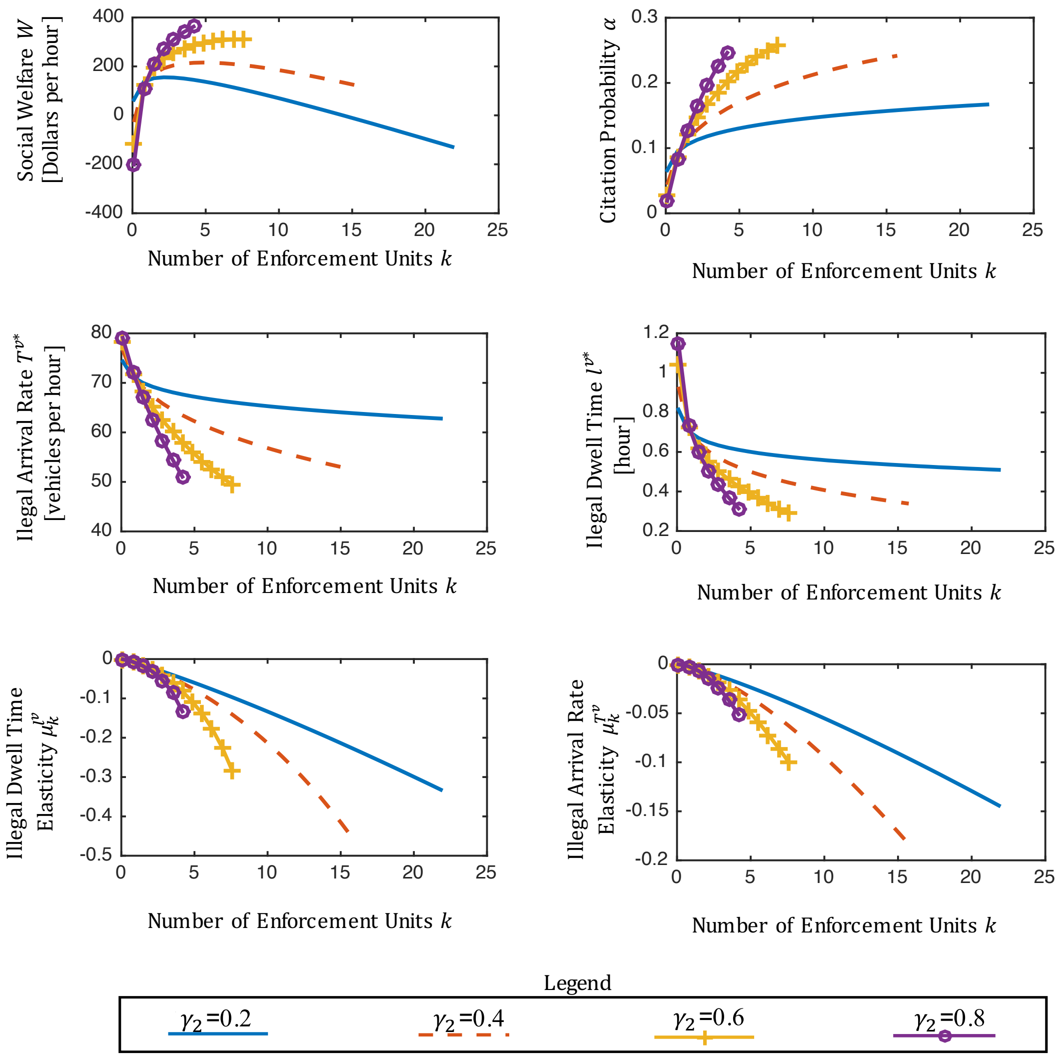

Fig. 8. Impact of the meeting elasticity γ_2 and level-of-enforcement k on social welfare.

7.3 Analysis of the dispersion parameter

We investigate the impact of the dispersion parameter on the optimal profit and social welfare. The results are illustrated in Fig. 9 and the following insights are observed. First, a higher dispersion parameter θ (i.e. a higher level of information in the system) is shown to decrease the optimal profit but increase the social welfare because fewer vehicles park illegally and get cited as they become aware of the higher cost of illegal parking. The conclusion from this observation is that cities should provide details of their parking enforcement plans if they wish to mitigate the negative externalities of illegal parking and improve social welfare. Second, increasing the elasticity γ_1 also improves social welfare at all values of θ. For profit, however, a different trend is observed: When θ is large, it is more profitable to have a large γ_1 and when θ is small, it more profitable to have a γ_1 that is neither too large nor too low.

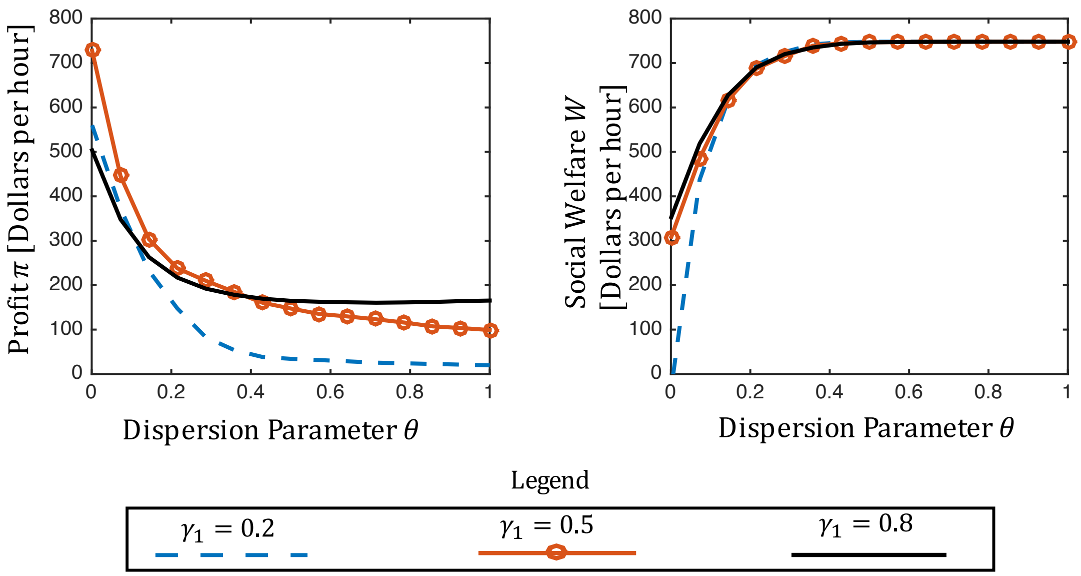

Fig. 9. Impact of the dispersion parameter θ on profit and social welfare.

8 Conclusions

We present an equilibrium model of illegal parking and use the model to devise optimal parking enforcement policies. By investigating the properties of the equilibrium model, we determine the interrelationship between the factors that affect illegal parking behavior and we illustrate these effects in Fig. 10. Our findings on the properties of the equilibrium model are summarized as the following:

1- The citation probability increases with the illegal dwell time because vehicles that are parked illegally for long time are more susceptible to receiving a citation. In contrast, the citation probability decreases with the illegal arrival rate because the enforcement units have more vehicles to inspect and cite.
2- The illegal utility is comprised of a benefit and a cost. Increasing the illegal dwell time raises both the benefit and the cost of illegal parking. The raise in benefit occurs because the vehicle enjoys a longer duration for completing a given activity. The raise in the cost, on the other hand, occurs because of the respective increase in the citation probability.
3- The meeting rate increases with the level-of-enforcement, illegal arrival rate, and illegal dwell time.
4- According to the first condition of equilibrium, the illegal dwell time is a function of the citation probability.
5- For a fixed dwell time, each vehicle is better off parking legally because of the larger obtained utility.

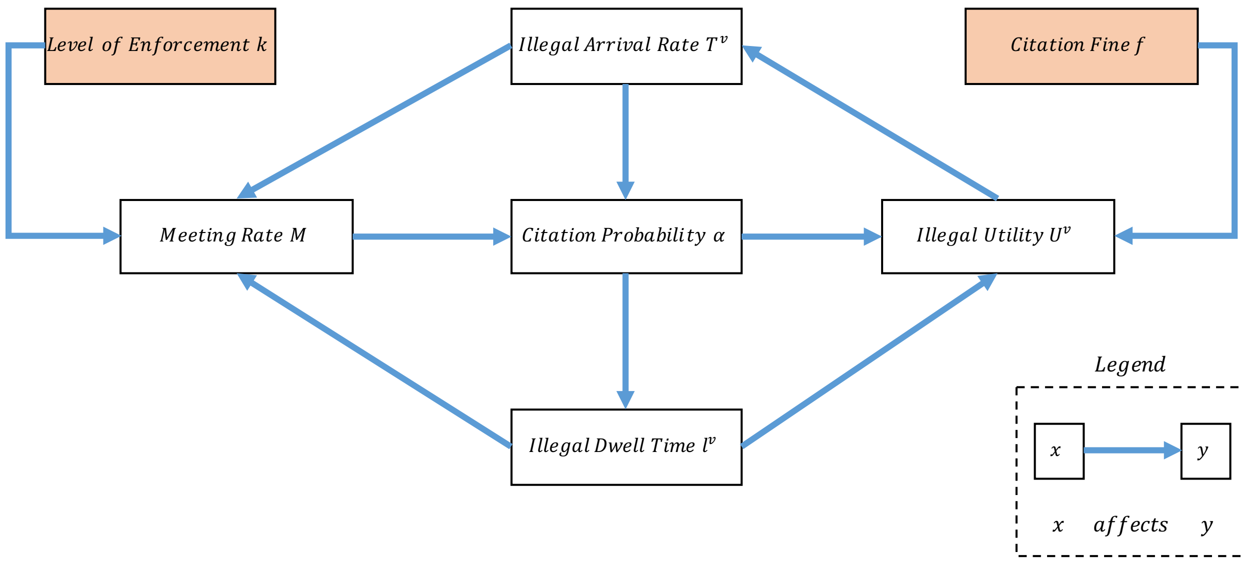

Fig. 10. Influential factors that affect illegal parking behavior.

Our findings on devising the optimal parking enforcement policy are summarized as follows:

1- When maximizing the profit, it is best to set the citation fine so large that some vehicles still park illegally but not large enough to deter illegal parking completely.
2- When maximizing social welfare, it is best set the fine so large such that no vehicle parks illegally and to have a small level-of-enforcement. This policy ensures that no vehicle parks illegally.
3- Profit cannot be simultaneously optimized with respect to the citation fine and the level-of-enforcement.

Despite the widespread application and ubiquity of parking enforcement policies in many cities, research in this area is still scarce and there is a need for studies that address the following extensions to the presented model. First, our model explicitly defines a policy as a given citation fine and level-of-enforcement. In reality, however, parking enforcement policies are multifaceted and include sub-policies that regulate parking through towing, issuing parking permits, or wheel clamping. A question that remains to be answered is when is it beneficial to introduce any of these sub-policies to parking enforcement and what is the effect of these sub-policies on social welfare and profit? As an example, a city that prioritizes social welfare is better off with towing illegally parked vehicles to take them off the street. Second, there is a need for optimizing the details of each of such sub-policies. As an example, if towing is a viable option, then what is the required number of towing trucks, enforcement units that find illegal vehicles, and the towing penalty paid by drivers? Similarly, if parking permits are the favorable sub-policy, then how many permits should be issued? Third, there are a number of assumptions made in this paper. Relaxing each of these assumptions, such as accounting for the parking search time of legal and illegal vehicles, is an avenue of future research.

Appendix A- Impact of the illegal dwell time and arrival rate on the citation probability

In this appendix, we describe the relationship between the citation probability with the illegal arrival rate and dwell time.

Lemma A1. The partial derivatives of the citation probability α with respect to illegal arrival rate T^v and dwell time l^v are defined respectively as $\partial\alpha/\partial T^v = \alpha \cdot (\gamma_1 - 1)/T^v$ and $\partial\alpha/\partial l^v = \alpha\gamma_1/l^v$.

Proof: The proof lies in taking the implicit differentiation of $\alpha = M(N^v, k)/T^v$ with respect to l^v which leads to the following

$$\frac{d\alpha}{dl^v} = [dM/dl^v \cdot T^v - dT^v/dl^v \cdot M]/(T^v)^2 \tag{50}$$

To simplify Eq. (50), we need to find $dM/dl^v = \partial M/\partial N^v \cdot dN^v/dl^v$ which is derived as

$$\frac{dM}{dl^v} = \alpha\gamma_1/l^v \cdot (dT^v/dl^v \cdot l^v + T^v) \tag{51}$$

Using Eq. (51) in Eq. (50), we can rewrite Eq. (50) as

$$d\alpha = \alpha \cdot (\gamma_1 - 1) \cdot dT^v/T^v + \alpha \cdot \gamma_1 \cdot dl^v/l^v \tag{52}$$

Lemma A1 can be readily proven from this result. ■

Appendix B- Proof of the existence of an equilibrium

In this appendix, we prove the existence of an equilibrium solution (l^{v^*}, T^{v^*}). The proof is involves showing that $\mathbf{T^v}$ is compact and convex (Lemma B1) and continuous (Lemma B3).

Lemma B1. The feasible set $\mathbf{T^v}$ is compact and convex when $U^v \geq U^n$.

Proof: We first prove that $\mathbf{T^v}$ is compact and second prove that it is convex. To prove compactness, we need to show that $\mathbf{T^v}$ is closed and bounded. We do this by proving that the illegal utility U^v is closed and bounded as well. Let us define the lower and upper boundaries of the illegal utility U^v as $\underline{U^v}$ and $\overline{U^v}$, respectively. The lower boundary can be defined at the very extreme case where an illegal vehicle receives no benefit of parking because of a short dwell time $l^v = 0$ but pays a fine of f as a result of getting cited right away. Hence, the lower boundary is defined as $\underline{U^v} = -f$. The upper boundary occurs at the other extreme end of the spectrum where an illegal vehicle parks for a very long time $l^v \to \infty$ but never gets cited which leads to a zero expected cost for illegal parking. This condition leads to a utility comprised strictly a benefit that is equal to $\overline{U^v} = \int_0^\infty s(l) \cdot dl$. Hence we have $\underline{U^v} \leq U^v \leq \overline{U^v}$ which shows that $\mathbf{T^v}$**,** a one-to-one logit function of U^v, is bounded and closed and hence compact. Next, we prove that that $\mathbf{T^v}$ is convex. This is a straightforward result when $U^v \geq U^n$ which indicates that only the convex part of the logit choice function is considered. ■

To show that $\mathbf{T^v}$ is continuous, we first present Lemma B2 as a prerequisite.

Lemma B2. The optimal dwell time $l^{v^*} = L(T^v)$ is unique for each $T^v \in [\underline{T^v}, T]$ where $\underline{T^v}$ is a lower bound on the illegal arrival rate.

Proof: To show that $l^{v*} = L(T^v)$ is unique, we first show that $L(T^v)$ exists for $T^v \in [\underline{T^v}, T]$. As indicated in the first condition of equilibrium (Eq. 17), the dwell time $L(T^v)$ is obtained as the solution of $s(l^{v*}) = \alpha(l^{v*}, T^v) \cdot f \cdot \gamma_1 / l^{v*}$; the two sides of this equation are plotted in Fig. B1. It is easy to show that the right-hand-side of this equation (i.e. $\alpha f \gamma_1 / l^v$) tends infinity when $l^v \to 0$, and it tends to zero as $l^v \to \infty$. As is illustrated, when $T^v > \underline{T^v}$, the two curves cross each other at two points, and when $T^v < \underline{T^v}$, the two curves do not cross each other. Hence, it is clear that $l^{v*} = L(T^v)$ has a solution as long as the two curves cross each other when the condition $T^v \in [\underline{T^v}, T]$ is satisfied.

We now show that $L(T^v)$ is unique. Given that $L(T^v)$ is obtained from the first-order optimality condition on U^v, to prove uniqueness it is sufficient to show that U^v is concave based on the second-order optimality condition. U^v, however, is not concave; the first term of U^v (the benefit of parking) is concave but the second term (the negative of the cost) is convex. Despite non-concavity of U^v with respect to l^v, we can still show that the solution is unique. Given that U^v has two components (benefit and cost) and given that each component is either strictly convex or strictly concave, there are at most two solutions that satisfy the first-order optimality condition. These two solutions are the points where the two curves cross each other in Fig. B1 and only the second solution (with a larger l^v) the is global maximum of U^v. Hence, only one of the two solutions is valid and $l^{v*} = L(T^v)$ is unique.■

Lemma B3. The feasible set $\mathbf{T^v}$ is continuous when $T^v \in [\underline{T^v}, T]$.

Proof: Continuity of $\mathbf{T^v}$ can be easily established based on the uniqueness proof presented in Lemma B2. ■

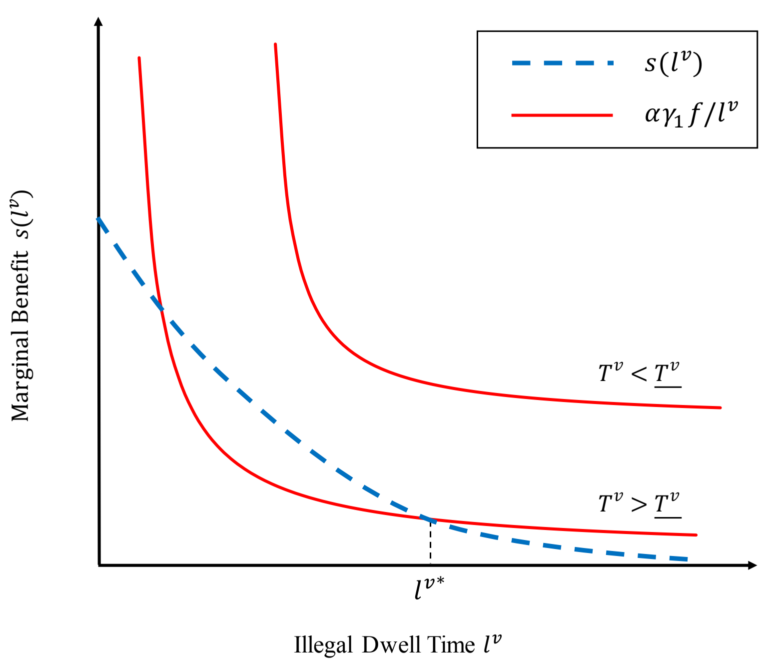

Fig. B1. Solution of the first condition of equilibrium.

Appendix C- Comparative static effects and elasticities

In this appendix, we present the comparative static effects of regulatory variables k and f on transitional variables and the interrelationship between the transitional variables.

We start with the citation fine f and investigate its impact on α, l^v, T^v, U^v, and M by taking the following derivatives.

$$\frac{dM}{df} = \frac{\partial M}{\partial N^v} \cdot \frac{dN^v}{df} \equiv \frac{\alpha\gamma_1}{N^v}\left(\frac{dl^v}{df}T^v + \frac{dT^v}{df}l^v\right) \tag{53}$$

$$\frac{d\alpha}{df} = \frac{d\alpha}{dl^v} \cdot \frac{dl^v}{df} + \frac{d\alpha}{dT^v} \cdot \frac{dT^v}{df} \tag{54}$$

$$\frac{dl^v}{df} = \gamma_1\left(\alpha + \frac{d\alpha}{df}f\right)/[s'(l^v)l^v + s(l^v)] \tag{55}$$

$$\frac{dT^v}{df} = \frac{\partial T^v}{\partial U^v} \cdot \frac{dU^v}{df} \equiv T^v\beta(1-\beta)\frac{dU^v}{df} \tag{56}$$

$$\frac{dU^v}{df} = s(l^v)\frac{dl^v}{df} - f\frac{d\alpha}{df} - \alpha \tag{57}$$

As is evident from the above conditions, we have five equations and five unknowns. Eq. (53) and (54) are straightforward. Eq. (55) is obtained from taking the implicit differentiation of Eq. (17), the first condition of equilibrium, with respect to l^v, Eq. (56) has the term $\partial T^v/\partial U^v$ which is obtained by taking the derivative of the logit choice model, and Eq. (57) is obtained by taking implicit differentiation of Eq. (4) with respect to f. Solving this system of equations yields the following two equations

$$\frac{dT^v}{df} = \frac{-(1-\beta)\theta f\alpha}{1+f\alpha\theta(1-\beta)(\gamma_1-1)} \tag{58}$$

$$\frac{dl^v}{df} = \frac{s(l^v)/[s'(l^v)l^v+s(l^v)(1-\gamma_1)]}{1+f\alpha\theta(1-\beta)(\gamma_1-1)} \tag{59}$$

which can be used to derive the following two elasticities

$$\mu_f^{T^v} = \frac{-(1-\beta)\theta f\alpha}{1+(1-\beta)(1-\gamma_1)\theta f\alpha} \tag{60}$$

$$\mu_f^{l^v} = \frac{s(l^v)/[s'(l^v)l^v+s(l^v)(1-\gamma_1)]}{1+(1-\beta)(1-\gamma_1)\theta f\alpha} \tag{61}$$

A very similar approach can be followed to obtain all other equations in Section 5.4.

Appendix D- Approximation of the optimal citation fine to maximize profits under a random choice structure

In this appendix, we prove that under a random choice structure (with $\theta = 0$), the citation fine f^* should be chosen so that illegal dwell time becomes a fixed value $l^v = -1/\delta$. An important result of this appendix is that the optimal citation fine f^* decreases with the technology parameter γ_1; the citation fine can be lowered when the inspection technology is efficient.

We proceed by (i) approximating the elasticity $\mu_f^{l^v}$ in Lemma D1, (ii) showing that the dwell time should be $l^v = -1/\delta$ in Lemma D2, and (iii) proving that the optimal citation fine f^* decreases with γ_1 in Lemma D3. The three lemmas are presented as follows.

Lemma D1. At the equilibrium solution, the elasticity of the illegal dwell time l^v with respect to the citation fine f can be approximated as $\mu_f^{l^v} = \frac{1}{\delta l^v + 1 - \gamma_1}$.

Proof: The proof of this lemma lies in our assumption that $s'(l)/s(l) = \delta, \forall l^v$ where δ is a constant. This assumption is justified when the marginal benefit function $s(l)$ is of the form $s(l) = B_0.(B_1)^l$ where B_0 and B_1 are constant parameters. With this assumption, and given that $\theta = 0$, we use Eq. (26) to obtain the approximation $\mu_f^{l^v} = 1/(\delta l^v + 1 - \gamma_1)$. ■

Lemma D2. Under a random choice structure (with $\theta = 0$), the citation fine f^* must be chosen such that the illegal dwell time is a fixed value equal to $l^v = -1/\delta$.

Proof: The proof can be easily obtained using the result of Lemma D1 and the optimality condition $\mu_f^{l^v} = -1/\gamma_1$ in Eq. (35). ■

Lemma D3. The optimal citation fine f^* decreases with γ_1.

Proof: Using the result of Lemma D2, we use Eq. (17) to calculate the optimal citation fine as

$$f^* = -\frac{s(-1/\delta)}{\gamma_1 \delta \alpha(0.5T, -1/\delta)} \tag{62}$$

where $\alpha(0.5T, -1/\delta)$, q, and $s(-1/\delta)$ are all fixed values. Hence, according to Eq. (62), f^* is inversely related to γ_1. ■

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